Study Guide for Linear Algebra – Exam 2

Term	Definition
Vector Space	A **Vector Space** is a nonempty set *V* of objects, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms (or rules) listed below. The axioms must hold for all vectors **u**, **v**, and **w** in *V* and for all scalars *c* and *d*. 1. The sum of **u** and **v**, denoted by **u + v**, is in *V*. 2. **u + v** = **v + u** 3. (**u + v**) + **w** = **u +** (**v + w**) 4. There is a **zero** vector **0** in *V* such that **u + 0** = **u** 5. For each **u** in *V*, there is a vector, **−u** in *V* such that **u** + (**−u**) = **0** 6. The scalar multiple of **u** by *c*, denoted by *c***u**, is in *V* 7. c(**u + v**) = *c***u** + *c***v** 8. (*c* + *d*)**u** = *c***u** + *d***v** 9. *c*(*d***u**) = (*cd*)**u** 10. 1**u** = **u**
Subspace	A **subspace** of a vector space *V*, is a subset *H* of *V* that has three properties: a. The zero vector of *V* is in *H* b. *H* is closed under vector addition. That is for **u** and **v** in *H*, the sum **u + v** is in *H* c. *H* is closed under multiplication by scalars. That is, for each **u** in *H* and each scalar *c*, the vector *c***u** is in *H*
Null space	The **null space** of an m x n matrix *A*, written as Nul *A*, is the set of all solutions to the homogeneous equation *A***x** = **0**. In set notation: Nul *A* = { **x** : **x** is in $\mathbf{R}^n$ and *A***x** = **0** }
Column space	The **column space** of an m x n matrix *A*, written Col *A*, is the set of all linear combinations of the columns of *A*. If *A* = [$\mathbf{a_1}$,… , $\mathbf{a_n}$], then Col *A* = Span{ $\mathbf{a_1}$,… , $\mathbf{a_n}$ }
Linear transformation	A **linear transformation** *T* from a vector space *V* into a vector space *W* is a rule that assigns to each vector **x** in *V* a unique vector *T*(**x**) in *W*, such that i. *T*(**u + v**) = *T*(**u**) + *T*(**v**) for all **u, v** in *V*, and ii. *T*(*c***u**) = *cT*(**u**) for all **u** in *V* and all scalars *c*

Linear independence	An indexed set of vectors { $\mathbf{v}_1$,… , $\mathbf{v}_p$ } is said to be **linearly independent** if the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = 0$ has only the trivial solution $c_1 = 0$,… , $c_p = 0$
Linear dependence	An indexed set of vectors { $\mathbf{v}_1$,… , $\mathbf{v}_p$ } is said to be **linearly dependent** if the vector equation $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_p\mathbf{v}_p = 0$ has a nontrivial solution, that is, if there are some weights, c_1 ,… , c_p, not all zero such that the equation holds …
Linear dependence relation	… In such a case (above), this is called a **linear dependence relation** among $\mathbf{v}_1$,… , $\mathbf{v}_p$
Basis	Let *H* be a subspace of a vector space *V*. An indexed set of vectors B = { $\mathbf{b}_1$,… , $\mathbf{b}_p$ } in *V* is a **basis** for *H* if i. *B* is a linearly independent set, and ii. the subspace spanned by *B* coincides with *H*; that is $H = \text{Span}\{ \mathbf{b}_1 ,\ldots, \mathbf{b}_p \}$
Coordinate vector	Suppose the set B = { $\mathbf{b}_1$,… , $\mathbf{b}_p$ } is a basis for *V* and **x** is in *V*. The **coordinates of x relative to the basis *B*** (or the ***B*−coordinates of x**) are the weights c_1 ,… , c_n such that $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$
Dimension	If *V* is spanned by a finite set, it is said to be finite−dimensional and the **dimension** of *V*, written as dim *V*, is the number of vectors in the basis for *V*. The dimension of the zero space { **0** }, is defined to be zero. If *V* is not spanned by a finite set, it is said to be infinite−dimensional
Rank	The **rank** of *A* is the dimension of the column space of *A*
Change of coordinates matrix	$P_B = [\ \mathbf{b}_1\ \mathbf{b}_2 \ldots\ \mathbf{b}_n\ ] = \mathbf{x} = P_B\ [\ \mathbf{x}\ ]_B$

Eigenvalue	A scalar λ is called an **eigenvalue** of *A* if there is a nontrivial solution **x** of $A\mathbf{x} = \lambda\mathbf{x}$; such an **x** is called an *eigenvector* corresponding to λ
Eigenvector	An **eigenvector** of an n x n matrix *A* is a nonzero vector **x** such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ
Eigenspace	The set of all solutions to: $(A - \lambda I)\mathbf{x} = 0$ **I** is the null space of $(A - \lambda I)$. This is called the **Eigenspace** of *A* corresponding to λ
Diagonalizable	A square matrix *A* is said to be **diagonalizable** if *A* is a similar to a diagonal matrix, that is: $A = PDP^{-1}$ for some invertible matrix *P* and some diagonal matrix *D*
Similar matrices	We say that *A* is **similar** to *B* if there is an invertible matrix *P* such that $PAP^{-1} = B$, or $A = PBP^{-1}$

Chapter.ThmNum	**theorem**
3.2	If *A* is a triangular matrix, the det *A* is the product of the entries on the main diagonal of *A*.
3.3 row ops	Let *A* be a square matrix a. If multiple of one row of *A* is added to another row to produce a matrix *B*, then det *A* = det *B* b. If two rows of *A* are interchanged to produce *B*, then det *B* = −det *A* c. If one row of *A* is multiplied by *k* to produce *B*, then det *B* = *k**det *A*
3.4	A square matrix *A* is invertible if and only if det *A* != 0
3.5	If *A* is an n x n matrix, then det A^T = det *A*

3.6 multiplicative property	If A and B are n x n matrices, then det AB = (det A)(det B)
3. Formula for det A as a product of pivots	det A $\begin{cases} (-1)^r * (\text{ product of pivots in } U \text{ }) \text{ ; when } A \text{ is invertible} \\ 0 \text{ ; when } A \text{ is not invertible} \end{cases}$
4.1	If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in a vector space V, then Span{ $\mathbf{v}_1, \dots, \mathbf{v}_p$ } is a subspace of V
4.2	The null space of an m x n matrix A is a subspace of $\mathbf{R}^n$, Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace or $\mathbf{R}^n$ **PROOF:** Nul A ***is*** a subset of $\mathbf{R}^n$ because A has n columns. Show that Nul A satisfies the three properties of subspaces: 1. Of course **0** is in Nul A − let **u** and **v** represent two vectors in Nul A, then $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$ 2. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$ (using a property of matrix multiplication), and thus closed under vector addition 3. $A(c\mathbf{u}) = c(A\mathbf{u}) = c(\mathbf{0}) = \mathbf{0}$ (where c is any scalar), which shows that $c\mathbf{u}$ is in Nul A, thus, Nul A is a subspace of $\mathbf{R}^n$
4.3	The column space of an m x n matrix A is a subspace of $\mathbf{R}^n$
4.4	An indexed set { $\mathbf{v}_1, \dots, \mathbf{v}_p$ } of two or more vectors, with $\mathbf{v}_1$!= 0, is linearly dependent if and only if some $\mathbf{v}_j$ (with j > 1) is a linear combination of the preceding vectors $\mathbf{v}_1, \dots, \mathbf{v}_{j-1}$
4.5 spanning set theorem	Let S = { $\mathbf{v}_1, \dots, \mathbf{v}_p$ } be a set in V and let H = Span{ $\mathbf{v}_1, \dots, \mathbf{v}_p$ }. a. If one of the vectors in S—say , $\mathbf{v}_k$—is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H b. If H != { **0** }, some subset of S is a basis for H
4.6	The pivot columns of a matrix A form a basis for Col A

4.7 unique representation theorem	Let B = { $\mathbf{b}_1$,… , $\mathbf{b}_n$ } be a basis for a vector space V. Then for each **x** in V, there exists a unique set of scalars c_1,… ,c_n such that $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ **PROOF:** Since B spans V, there exists scalars such that $\mathbf{x} = c_1\mathbf{b}_1 + \ldots + c_n\mathbf{b}_n$ holds true. Suppose **x** also has the representation: $\mathbf{x} = d_1\mathbf{b}_1 + \ldots + d_n\mathbf{b}_n$ for scalars d_1 ,… , d_n. Then subtracting, we have: $\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \ldots + (c_n - d_n)\mathbf{b}_n$ Since B is linearly independent, the weights in the second equation must all be zero. That is, $c_j = d_j$ for 1 <= j <= n
4.8	Let B = { $\mathbf{b}_1$,… , $\mathbf{b}_n$ } be a basis for a vector space V. Then the coordinate mapping **x** |−> [**x**]$_B$ is a one−to−one linear transformation from V onto $\mathbf{R}^n$.
4.9	The vector space V has a basis B = { $\mathbf{b}_1$,… , $\mathbf{b}_n$ }, then any set in V containing more than n vectors must be linearly dependent
4.10	If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors
4.12 Basis Theorem	Let V be a p−dimensional vector space, p >= 1. Any linearly independent set of exactly p elements in V is automatically a basis for V. Any set of exactly p elements that spans V is automatically a basis for V.
4.13	If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as B
4.14 The rank theorem	The dimensions of the column space and the row space of an m x n matrix A are equal. This common dimension, the rank of A, also equals the number of pivot positions in A and satisfies the equation: rank A + dim Nul $A = n$

IMT – cont'd	Let *A* be an m x n matrix. Then the following statements are each equivalent to the statement that *A* is an invertible matrix. m. The columns of *A* form a basis for $\mathbf{R}^n$. n. Col $A = \mathbf{R}^n$ o. dim Col $A = n$ p. rank $A = n$ q. Nul $A = \{ \mathbf{0} \}$ r. dim Nul $A = 0$
5.1	The eigenvalues of a triangular matrix are the entries on its main diagonal.
IMT – the END!!	Let *A* be an n x n matrix. Then *A* is invertible if and only if: s. The number 0 is *not* an eigenvalue of *A*
5.2	If $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an n x n matrix *A,* then the set $\{ \mathbf{v}_1, \ldots, \mathbf{v}_r \}$ is linearly independent
5.3 Properties of determinants	Let *A* and *B* be n x n matrices. a. *A* is invertible if and only if det *A* != 0 b. det AB = (det A)(det B) c. det A^T = det A d. If *A* is triangular, then det *A* is the product of the entries on the main diagonal of *A* e. A row replacement operation on *A* does not change the determinant. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same as the scalar factor
5.4	If n x n matrices *A* and *B* are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities)
5.5 The digitalization theorem	An n x n matrix *A* is diagonalizable if and only if *A* has *n* linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with *D* a diagonal matrix, if and only if the columns of *P* are *n* linearly independent eigenvectors of *A*. In this case, the diagonal entries of *D* are the eigenvalues of *A* that correspond, respectively, to the eigenvectors in *P*.
5.6	An n x n matrix with *n* distinct eigenvalues is diagonalizable

5.7	Let *A* be an m x n matrix whose distinct eigenvalues are λ_1 ,… , λ_p a. For 1 <= k <= p, the dimension of the Eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k b. The matrix *A* is diagonalizable if and only if the sum of the dimensions of the distinct eigenspaces equals *n*, and this happens if and only if the dimension of the eigenspace for each λ_k equals the multiplicity of λ_k c. If *A* is diagonalizable and B_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets B_1 ,… , B_1 forms an eigenvector basis for $\mathbf{R}^n$